

# Local Dynamics of Periodic Point

Thm. (Koenigs Linearization) Suppose  $f(z_0) = z_0$ ,  $0 < |\lambda| < 1$ .

1) Suppose  $0 < |\lambda| < 1$  (gen. attr.). Then  $\exists U \ni z_0$  open set  $f: U \rightarrow \mathbb{C}$  homeom. + is cont. its inv. s.t.



$$\begin{array}{ccc} U & \xrightarrow{f} & U \\ \downarrow \phi & & \downarrow \phi \\ \mathbb{C} & \xrightarrow{\cdot \lambda} & \mathbb{C} \end{array} \quad \phi(f(z)) = \lambda \phi(z)$$

2) Suppose  $|\lambda| > 1$ . Then  $\exists U \ni z_0$  open set  $f: U \rightarrow \mathbb{C}$  hnt + inv cont. its inv. s.t.

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{f} & U \\ \downarrow \phi & & \downarrow \phi \\ \mathbb{C} & \xrightarrow{\cdot \lambda} & \mathbb{C} \end{array}$$

Proof.  $f(0) = 0$ .  $\phi(z) := \lim_{n \rightarrow \infty} \frac{f^n(z)}{\lambda^n}$

A)

$$1) \phi(f(z)) = \lim_{n \rightarrow \infty} \frac{f^{n+1}(z)}{\lambda^n} = \lambda \cdot \lim_{n \rightarrow \infty} \frac{f^{n+1}(z)}{\lambda^{n+1}} = \lambda \phi(z).$$

$$2) \exists c > 1 \text{ s.t. } c^2 < |\lambda| < c, \text{ and } r > 0 \text{ s.t.}$$

$$|f(z)| \leq c|z| \text{ for } |z| < r.$$

$$3) \exists M > 1 \text{ s.t. } |f(z) - \lambda z| \leq M|z|^2$$

$$|f^{n+1}(z) - \lambda f^n(z)| \leq M r^2 c^{2n}$$

$$4) \left| \frac{f^{n+1}(z)}{\lambda^{n+1}} - \frac{f^n(z)}{\lambda^n} \right| \leq \frac{M r^2 c^{2n}}{|\lambda|^n} = M r^2 \left( \frac{c^2}{|\lambda|} \right)^n \rightarrow 0$$

$\Rightarrow$  seq. hnt to hnt  $\phi$

5)  $f'(0) = \lim_{n \rightarrow \infty} \frac{(f^n)'(0)}{\lambda^n} = 1$ , so locally  $f$  is an open  $U \ni 0$ .

B)  $\lambda \neq 0$ , so  $f$  is locally invertible. Choose  $V$  s.t.  
 $f|_V: V \xrightarrow{\sim} f(V)$ . Apply part (A) to  $f^{-1}$ .

### Cor. (Global Linearization)

A)  $0 < |\lambda| < 1$ :  $f$  extends to a holomorphic  $A(z_0) \rightarrow \mathbb{C}$  s.t.

$$A(z_0) \xrightarrow{f} A(z_0)$$

$$\begin{array}{ccc} f & & \\ \downarrow & \cdot \lambda & \downarrow \\ \mathbb{C} & \xrightarrow{\quad} & \mathbb{C} \end{array}$$

B)  $|\lambda| > 1$ :  $f = f^{-1}$  extends to a holomorphic  $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  s.t.

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} \\ \uparrow f & & \uparrow f \\ \mathbb{C} & \xrightarrow{\cdot \lambda} & \mathbb{C} \end{array}$$

Proof. A) If  $z \in A(z_0)$ , then  $f^n(z) \in U$ . Define  
 $f(z) = \lambda^{-n} f(\lambda^n z)$ .

B) If  $z \in \mathbb{C}$ ,  $\lambda^{-n} z \in f(U)$ . Define  $f(z) = \lambda^n f(\lambda^{-n} z)$ .

Cor. If  $f$  has a rep. per. pt. and  $\lambda \notin \mathbb{R}$ , then  
 $J(f)$  is not smooth.

Proof. Since 0 is not isolated, there  $z_0 \in J \cap U$  lie  
along the axis where  $f^{-1}$  is a log. spiral.

Thm. The attracting basin of a periodic cycle  
contains a critical point.

Cor. There are at most  $2d-2$  attracting cycles for  $f$  of degree  $d$ .

Proof of Thm.  $f(z_0) = z_0$ .

$A_0$  = component of  $A$  containing  $z_0$  immediate basin

HLW A proper holomorphic map with no critical point is a covering map.

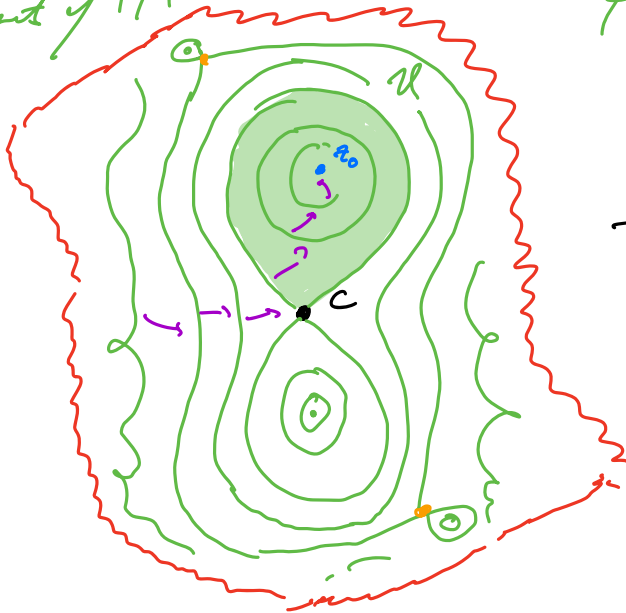
• If there is no cent pt, then  $f: A_0 \rightarrow A_0$  is a covering map.  $A_0$  is hyperbolic.

By Pich,  $f$  is a hyp. isometry, which contradicts the fact that  $f$  is attracting on  $A$ .

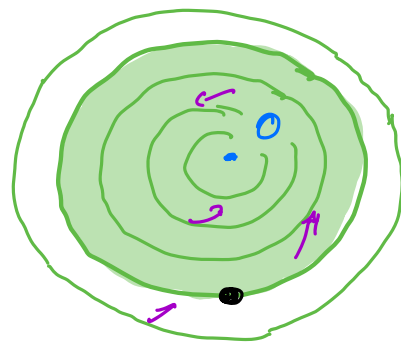
Picture of the attracting basin:

Level set of  $|f|$

$f$  is iso on  $U$



$f$



Cent pt attracts  $f^{-1}$

# Superattracting Cycles

Thm. (Böttcher conjugacy) Suppose  $f(z_0) = z_0$ ,

$\lambda = f'(z_0) = 0$ , and the cent. pt. is of degree  $n$ . Then

$\exists U \ni z_0$  open and  $\phi: U \rightarrow \mathbb{C}$  hsb st.

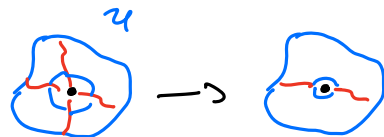
$\phi: U \xrightarrow{\sim} \phi(U) \subset \mathbb{D}$  and

$$U \xrightarrow{f} U$$

$$\phi \downarrow \qquad \qquad \downarrow \phi$$

$$\mathbb{C} \longrightarrow \mathbb{C}$$

$$w \longmapsto w^n$$



Proof. 1)  $f'(0) = 0$ ,  $f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots$

Pick  $c$  st.  $c^{n-1} = a_n$ .  $c f(z/c) = z^n (1 + b_1 z + \dots)$ .

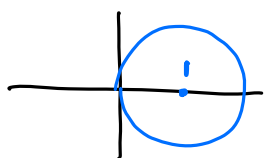
Set  $f(z) = z^n (1 + b_1 z + \dots) = z^n (1 + g(z))$

2) Koenig:  $\lambda \phi(z) = \phi(f(z)) \rightarrow \phi(z) = \lim_{k \rightarrow \infty} \frac{f^k(z)}{\lambda^k}$

Want:  $\phi(z)^n = \phi(f(z)) \rightarrow \phi(z) = \lim_{k \rightarrow \infty} \sqrt[n]{f^n(z)}$  *Satisfies relation!*

$$f^k(z)^{1/n^k} = \prod_{i=1}^k \frac{f^i(z)^{1/n^i}}{f^{i-1}(z)^{1/n^{i-1}}} = \prod_{i=1}^k \frac{f(f^{i-1}(z))^{1/n^i}}{f^{i-1}(z)^{1/n^{i-1}}} = \prod_{i=1}^k (1 + g(f^{i-1}(z)))^{1/n^i}$$

For  $|z| < 1$ ,



$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2} z^2 + \dots$$

always exists for all  $\alpha$ .  $\alpha = \frac{1}{n^i}$

Set  $\phi_n(z)$  is the product.

• Pick  $r_1, C > 0$  s.t.  $|g(z)| \leq C|z|$  for  $|z| < r_1$ .

• Let  $r_2 > 0$  be s.t.  $r_2^{n-1}(1+Cr_2) = 1$ .

• Let  $r = \min(r_1, r_2, 1/(2C))$ .  $\forall z$   $|z| < r$ ,

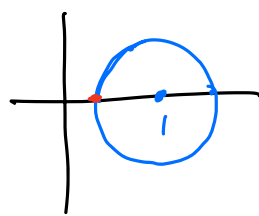
$$|f(z)| \leq |z|^n |1+g(z)| \leq |z| \underbrace{r^{n-1}(1+Cr)}_{< 1} < |z|$$

$$\Rightarrow |f^k(z)| < r$$

$$|g(f^{i-1}(z))| \leq Cr \leq \frac{1}{2} \Rightarrow (1+g(f^{i-1}(z)))^{\frac{1}{n^i}}$$

3)  $f_n$  converges if  $\sum_{i=1}^{\infty} \frac{\log |1+g(f^{i-1}(z))|}{n^i}$  converges. <sup>split</sup>

\*  $|\log 1+w|$  has a max on  $\{|w| \leq \frac{1}{2}\}$  at  $w = -\frac{1}{2}$



$$\frac{\log |1+g(f^{i-1}(z))|}{n^i} \leq \frac{|\log 1+g(f^{i-1}(z))|}{n^i} \leq \frac{\log 2}{n^i} \text{ inf.}$$

• Riemann map extended to  $A \rightarrow \mathbb{C}$  because  $\lambda^{-1}$  makes sense. But  $\sqrt{\lambda}$  does not always make sense.

Prop. A) There is a maximal  $0 < r < 1$  for which

$\psi = \phi^{-1}: D_r \rightarrow A_0(z_0)$  exists and is biholomorphic.

If  $r = 1$ , then  $\psi: D_r \xrightarrow{\sim} A_0(z_0)$ . \* Thm!

If  $r < 1$ , then there is a critical point on

$$\partial \psi(D_r) \subset A_0(z_0).$$

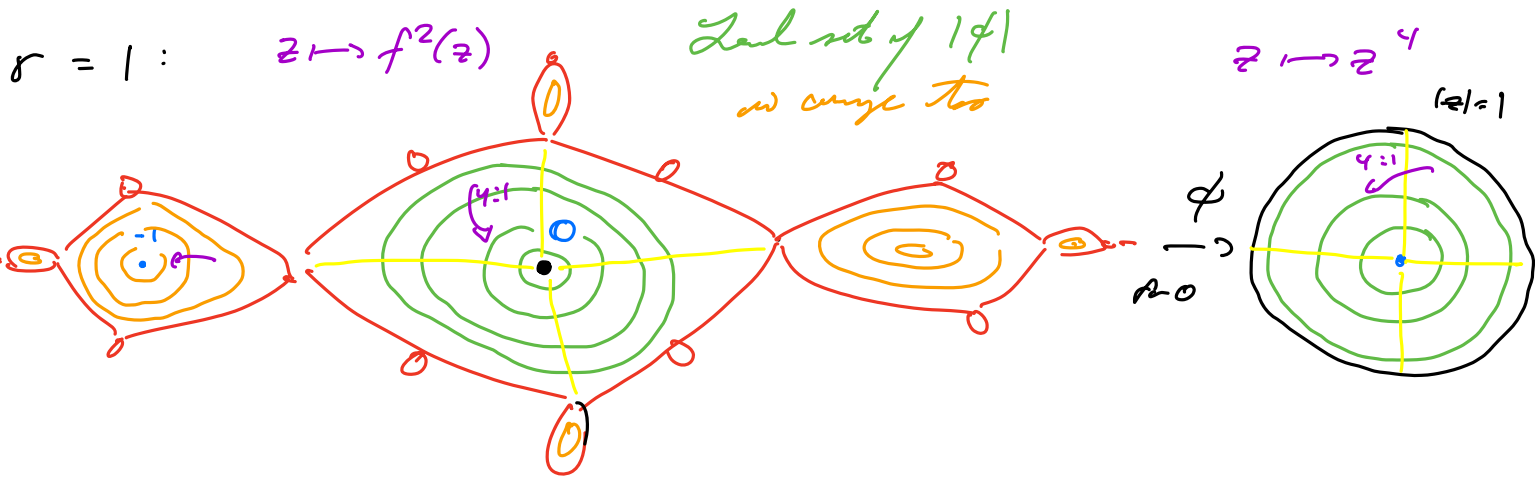
potential

Green's function

B)  $|f|$  extends to a cont. funcn  $A(z_0) \rightarrow [0, \infty)$

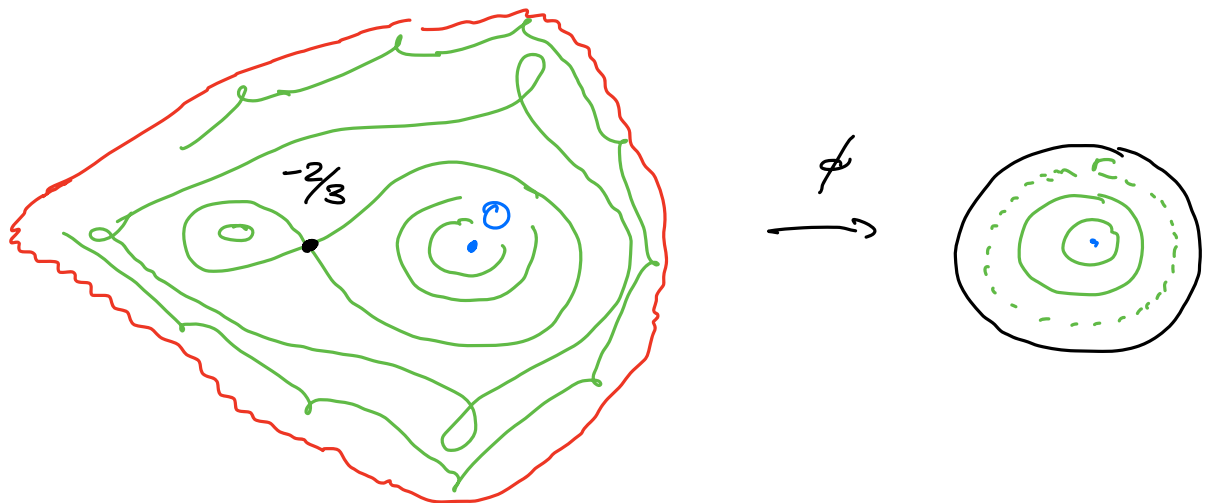
Proof of B) If  $z \in A$ ,  $|f|(z) := |f(f^k(z))|^{1/k}$  for  $f^k(z) \in \mathcal{U}$ .

Ex: Blaschke  $f(z) = z^2 - 1$ ,  $0 \mapsto -1 \mapsto 0$



Ex:  $f(z) = z^3 + z^2$ ,  $0 \mapsto 0$ , end at  $-2/3$

$r < 1$ :



Proof of A) Let  $r$  be the max st.

$\psi: D_r \rightarrow A(z)$  exists.

1)  $\psi$  has no crit. pts in  $D_r$ :  $\psi'(w_0) = 0$   
 $w_0 \neq 0$

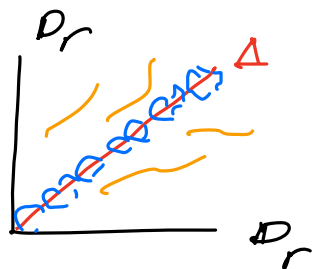
$$\psi(w^n) = f(\psi(w))$$

$$\Rightarrow n(\psi^{-1})'(w^n) w^{n-1} = f'(\psi(w)) (\psi^{-1})'(w)$$

$$\Rightarrow (\psi^{-1})'(w^n) = 0 \quad \leadsto w_0, w_0^4, w_0^{n^2}, \dots$$

all crit pts.

2)  $\{(w_1, w_2) \in \mathbb{D}_r \times \mathbb{D}_r \mid \psi^{-1}(w_1) = \psi^{-1}(w_2), w_1 \neq w_2\}$  is closed in  $\mathbb{D}_r \times \mathbb{D}_r$ : no crit pts  $\Rightarrow \psi^{-1}$  is loc. inj.



3)  $\psi^{-1}$  is injective:

Suppose  $\psi^{-1}(w_1) = \psi^{-1}(w_2)$ .  $|\psi|(\psi^{-1}(w_1)) = |w_1| = |w_2|$ . Pick  $(w_1, w_2)$  st.  $|w_1| = |w_2|$  is minimal. But since  $\psi^{-1}$  is open, for  $w_1'$  near  $w_1$ ,  $|w_1'| < |w_1|$ ,  $\exists w_2'$  near  $w_2$  st.  $\psi^{-1}(w_1') = \psi^{-1}(w_2')$ .

4)  $r=1$ : If  $\psi^{-1}(\mathbb{D}) \not\subset A_0$ , then  $\exists z' \in \partial \psi^{-1}(\mathbb{D})$ ,  $w_i \in \mathbb{D}$ ,  $\psi^{-1}(w_i) \rightarrow z'$ .  $|\psi|(\psi^{-1}(w_i)) = |w_i| \rightarrow 1$ , so  $|\psi|(z') = 1$ . But then  $f(z') \rightarrow 0$ .

5)  $r < 1$ :  $2\psi^{-1}(\mathbb{D}_r) \subset A_0$ :

$$f(\overline{\psi^{-1}(\mathbb{D}_r)}) \subset \overline{f(\psi^{-1}(\mathbb{D}_r))} \subset \psi^{-1}(\mathbb{D}_{r^2}) \subset A_0, \text{ and}$$

the backward invariance of  $A_0$ .

6)  $r < 1$ :  $|w'| = r$ ,  $\psi^{-1}(w_i) \rightarrow z' \in \partial \psi^{-1}(\mathbb{D}_r)$ ,  $w_i \rightarrow w$ . If  $z'$  is not a crit. pt of  $f$ , then

a branch  $f^{-1}$  split locally, and we can set  
 $\phi^{-1}(w) = f^{-1}(\phi^{-1}(w^n))$  near  $w'$ .  
 Contradicts simplicity.

